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Equidistribution of Random Walks on Spheres

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We study the equidistribution on spheres of the n-step transition probabilities of random walks on graphs. We give sufficient conditions for this property being satisfied and for the weaker property of asymptotical equidistribution. We analyze the asymptotical behaviour of the Green function of the simple random walk on $\mathbb{Z}^2$ and we provide a class of random walks on Cayley graphs of groups, whose transition probabilities are not even asymptotically equidistributed.

KEY WORDS: Markov chain; nearest neighbour type; Cayley graph; Green function; isotropy; asymptotical isotropy.

1. INTRODUCTION

In this paper we analyze some properties related to the distribution of the n-step transition probabilities of random walks. In particular, if not otherwise explicitly stated, we consider irreducible random walks of *nearest neighbour* type on *locally finite* graphs, that is random walks (X, P) where X is the vertex set of a connected graph, $P = (p(x, y))_{x, y \in X}$ is the stochastic transition matrix which describes the one-step transitions of a Markov chain $\{Z_n\}_{n \in \mathbb{N}}$, with state space X and P has the property that $p(x, y) > 0$ if and only if x and y are neighbours. The standard example of a random walk of nearest neighbour type is the *simple random walk*, that is, if $\deg(x)$ is the number of neighbours of x, then $p(x, y) = 1/\deg(x)$ if x and y are neighbours, and 0 otherwise.

We first consider (Paragraph 2) equidistribution of the n-step transition probabilities $p^{(n)}(x, y)$ on the spheres $S(x, k) = \{ y \in X : d(x, y) = k\}$ (where d is the natural distance on the graph).

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Definition 1.1. A random walk (X, P) is said to be *isotropic* with respect to a point $x_0 \in X$ if for every fixed $k \in \mathbb{N}$, $n \in \mathbb{N}$ and for all $y_1, y_2 \in S(x_0, k)$, we have that $p^{(n)}(x_0, y_1) = p^{(n)}(x_0, y_2)$.

Note that this definition can be given also in terms of *Green functions* (see Eq. (9)).

Particular symmetric graphs, such as for example the free product of n copies of a complete graph with a finite number of vertices or the homogeneous tree are *radial* with respect to (at least) one of their vertices x_0 (we will be more specific in Paragraph 2). Isotropy is strongly linked with the action of the automorphism group Γ of X. Every Γ_{x_0}-*invariant* random walk (in particular the simple random walk) on an x_0-radial graph is isotropic with respect to x_0 (Theorem 2.2). Moreover, we classify the isotropic random walks on the trees which are radial with respect to one of their vertices (Proposition 2.3).

When equidistribution is not satisfied, we look for some kind of asymptotic equidistribution (Paragraph 3) of the transition probabilities according to the following definition.

Definition 1.2. The random walk is said to be *asymptotically isotropic* with respect to a point $x_0 \in X$ if for every fixed $k \in \mathbb{N}$, and for all $y_1, y_2 \in S(x_0, k)$ we have that $p^{(n)}(x, y_1) \sim^n p^{(n)}(x, y_2)$.

We recall that if $\{a_n\}_n$ and $\{b_n\}_n$ are two sequences then $a_n \sim^n b_n$ if there exists a sequence $\{o_n\}_n$ and $n_0 \in \mathbb{N}$ such that, $\forall n \geqslant n_0$, $a_n = b_n(1 + o_n)$, and $\lim_{n\to\infty} o_n = 0$.

Given a finitely generated, discrete group Γ and an irreducible probability measure μ on Γ (that is, the support of μ generates Γ as a semigroup), we denote by (Γ, μ) the random walk on the Cayley graph of Γ and with transition probabilities

$$p(x, y) = \mu(x^{-1}y), \qquad x, y \in \Gamma$$

In Paragraph 3 we prove a necessary and sufficient condition for the asymptotical isotropy of a recurrent, strongly periodic (or strongly aperiodic) random walk (Theorem 3.2). In particular this condition is satisfied by every recurrent random walk on the Cayley graph of a group (Remark 3.3).

We also extend a result obtained by Avez in (ref. 1, Theorem 1) to the general case of symmetric random walks on amenable groups (Theorem 3.4) and such a result implies that these random walks are asymptotically isotropic.

Paragraph 4 is devoted to the study of the Green function $G_{\mathbb{Z}^2}((0,0),(a,b)\,|\,z)$ of the simple random walk on $\mathbb{Z}^2$ (see Eq. (18)). This study is needed for the estimates in Paragraph 5, but is also interesting in itself. We first obtain an expression for the n-step transition probabilities and we estimate them for n going to infinity, deriving such facts from the transition probabilities of the simple random walk on $\mathbb{Z}$ (Corollary 4.3). Then, using Laplace-type techniques, we obtain an asymptotic estimate for $G_{\mathbb{Z}^2}((0,0),(a,b)\,|\,z)$ when (a,b) goes to infinity along a straight line, and for any $z \in (0,1)$ (Theorem 4.5).

In Paragraph 5 we study the asymptotic behaviour of the transition probabilities on the free product of the simple random walk on $\mathbb{Z}^2$ and a random walk on a group Γ. First we introduce some technical instruments developed by Woess,(2) Cartwright and Soardi,(3) see also Cartwright;(4) then we choose two points on a sphere $S(e,k)$ (where e is the unit element of the free product) and we compare the asymptotic values of the transition probabilities from e to each point. We prove that, under rather general conditions (Assumption 5.1) the random walk on the free product is far from being asymptotically isotropic (Theorem 5.2).

This provides us a whole family of random walks which, despite their properties of symmetry, are not asymptotically isotropic, including the simple random walk on $\mathbb{Z} * \mathbb{Z}^2$. This is the simplest among the *locally free groups* which were treated by Nechaev, Grosberg and Vershik,(5) Paragraph 3. Their Theorem 5 gives an estimate of the asymptotic values of the transition probabilities assuming that they are equidistributed on spheres, which contrasts our rigorous computations.

2. ISOTROPIC RANDOM WALKS

Let us recall that, if (X,d) is a graph with its natural distance, $\mathrm{AUT}(X)$ is the group of all the bijective maps γ from X onto itself such that $x \sim y$ if and only if $\gamma x \sim \gamma y$, where $\sim$ denotes neighbourhood. We note that $\gamma \in \mathrm{AUT}(X)$ if and only if γ is an isometry from X onto itself; in particular for all $x \in X$ and all $k \in \mathbb{N}$, γ is an isometry from $S(x,k)$ onto $S(\gamma x,k)$ and $\deg(x) = \deg(\gamma x)$.

Let Γ be a subgroup of $\mathrm{AUT}(X)$: a random walk (X,P) is called *invariant* if for all $x, y \in X$, for all $\gamma \in \Gamma$ we have that $p(x,y) = p(\gamma x, \gamma y)$.

Definition 2.1. A graph (X,d) is called x-*radial* if $\Gamma_x = \{\gamma \in \mathrm{AUT}(X) : \gamma x = x\}$ acts transitively on $S(x,k)$ for all $k \in \mathbb{N}$.

A graph (X,d) is called *distance transitive* if for all $x, y, x', y' \in X$ such that $d(x,y) = d(x',y')$ there exists $\gamma \in \mathrm{AUT}(X)$ satisfying $\gamma x = x'$, $\gamma y = y'$.

We observe that a distance transitive graph is also radial with respect to every $x \in X$, but the converse is not true (see Proposition 2.3 (ii)). If X is x-radial then $\deg(y)$ depends only on $d(x, y)$; in particular, if X is distance transitive, $\deg(\cdot)$ is constant. Moreover if (X, P) is a Γ_x-invariant random walk and X is an x-radial graph then X must be locally finite.

It is worth noting that there is a complete characterization of all infinite, locally finite, distance transitive graphs X, that is $X = \underbrace{\mathbb{K}_r * \cdots * \mathbb{K}_r}_{n \text{ times}}$, i.e., the free product of n copies of the complete graph with r vertices $\mathbb{K}_r$ (see Macpherson[6] and Ivanov,[7] Theorem 4).

Theorem 2.2. (i) Let (X, P) be a Γ_{x_0}-invariant random walk (where $x_0 \in X$) on an x_0-radial graph X, then (X, P) is isotropic with respect to x_0.

(ii) Let X be a tree; if (X, P) is a random walk isotropic with respect to $x_0 \in X$ then (X, P) is Γ_{x_0}-invariant.

Proof. (i) Let $k \in \mathbb{N}$, $y_1, y_2 \in S(x_0, k)$ and $\gamma \in \Gamma_{x_0}$ such that $\gamma y_1 = y_2$. Let $x, y \in X$,

$$p^{(n)}(x, y) = \sum_{x_1, \dots, x_{n-1} \in X} p(x, x_1)\, p(x_1, x_2) \cdots p(x_{n-1}, y) = p^{(n)}(\gamma x, \gamma y)$$

where in the last equality we applied the hypothesis of Γ_{x_0}-invariance and the bijectivity of γ. If we set $x = x_0$, $y = y_1$ we get the thesis.

(ii) We first note that on a tree every vertex $x \in S(x_0, k)$, $k \geqslant 1$, has exactly one neighbour on $S(x_0, k-1)$ and all the others on $S(x_0, k+1)$. Let $\gamma \in \Gamma_{x_0}$, we have that

$$p(x, y) > 0 \Leftrightarrow x \sim y \Leftrightarrow \gamma x \sim \gamma y \Leftrightarrow p(\gamma x, \gamma y) > 0$$

Then let us take $x \sim y$; if $x = x_0$, the property of isotropy implies that $p(x_0, y) = p(\gamma x_0, \gamma y)$. If $x \neq x_0$ and $k = d(x, x_0)$, we have two cases:

(a) $y \in S(x_0, k+1)$, then

$$p(x, y) = \frac{p^{(k+1)}(x_0, y)}{p^{(k)}(x_0, x)} = \frac{p^{(k+1)}(x_0, \gamma y)}{p^{(k)}(x_0, \gamma x)} = p(\gamma x, \gamma y)$$

(b) $y \in S(x_0, k-1)$, then

$$p(x, y) = 1 - \sum_{w \neq y} p(x, w) = 1 - \sum_{w \neq y} p(\gamma x, \gamma w) = p(\gamma x, \gamma y) \qquad \blacksquare$$

This result can be applied to describe the isotropy of the random walks on a class of trees. Given a sequence $\{n_k\}_k \subset \mathbb{N}$, $k = 0, 1, 2,...$, we denote by $T_{\{n_k\}}$ the tree branching from a vertex o such that if $d(o, x) = k$ then $\deg(x) = n_k$.

Proposition 2.3. (i) A tree X is o-radial if and only if $X = T_{\{n_k\}}$ for some $\{n_k\}$; hence a random walk on $T_{\{n_k\}}$ is isotropic with respect to o if and only if it is invariant.

(ii) $T_{\{n_k\}}$ is radial with respect to any of its vertices x if and only if $n_k = n_{k+2}$ for all $k = 0, 1,...$, in particular it is distance transitive if and only if n_k is constant; under these hypotheses a random walk on $T_{\{n_k\}}$ is isotropic with respect to x if and only if it is Γ_x-invariant.

Proof. (i) The proof is very easy (remember that if X is o-radial then $\deg(\cdot)$ is constant on $S(o, k)$ for every $k \in \mathbb{N}$. The characterization of all the random walks which are isotropic with respect to o follows from Theorem 2.2.

(ii) The proof is straightforward and we omit it. ∎

Local limit theorems and Green kernel asymptotics for isotropic random walks on a homogeneous tree were derived by Sawyer (see ref. 8).

3. ASYMPTOTICALLY ISOTROPIC RANDOM WALKS

In this section we want to obtain a class of asymptotically isotropic (but in general not isotropic) random walks. Suppose that (X, P) is a periodic random walk with period d; (X, P) is called *strongly periodic* if there exists $n_0 \in \mathbb{N}$ such that $\inf_{x \in X} p^{(nd)}(x, x) > 0$, for all $n \geqslant n_0$ (we include in this definition the case $d = 1$, which is usually called *strongly aperiodic*). We note that strong periodicity holds for all random walks which are invariant under a *quasi-transitive* group action (see ref. 9, Paragraphs 2.D and 5.A), in particular for random walks on groups.

A random walk (X, P) is called *reversible* if there exists a strictly positive function v on X such that $v(x)\, p(x, y) = v(y)\, p(y, x)$, for all $x, y \in X$; we note that the period of such a random walk must be equal to 1 or 2.

In the following theorem the random walk is not required to be of nearest neighbour type.

Theorem 3.1. Let (X, P) be a recurrent, strongly periodic random walk of period d; then for all $x, y \in X$ we have

$$p^{(n+m)}(x, y) \sim^n p^{(n)}(y, y) \tag{1}$$

where $m \in \mathbb{N}$ such that $p^{(m)}(x, y) > 0$.

Proof. We note that it is enough to consider the case $n \equiv 0 \pmod{d}$, in fact in the other cases the probabilities involved in Eq. (1) are 0.

Let us fix $y \in X$ and let $C_0, C_1, \ldots, C_{d-1}$ be the periodicity classes such that $y \in C_0$. For every $x \in X$ we define k_x such that $x \in C_{d-k_x}$. We observe that if (X, P) is recurrent and strongly periodic by (ref. 9, Theorem 5.2(b))

$$\lim_{n \to +\infty} \frac{p^{((n+1)\,d+k_x)}(x, y)}{p^{(nd+k_x)}(x, y)} = 1 \tag{2}$$

Let $n_x \in \mathbb{N}$ such that $p^{((n_x+1)\,d-k_x)}(y, x) > 0$ and $\alpha_x = (p^{((n_x+1)\,d-k_x)}(y, x))^{-1}$, then by Eq. (2)

$$\begin{aligned} \frac{p^{(nd+k_x)}(x, y)}{p^{(nd)}(y, y)} &= \alpha_x \frac{p^{((n_x+1)\,d-k_x)}(y, x)\; p^{(nd+k_x)}(x, y)}{p^{(nd)}(y, y)} \\ &\leqslant \alpha_x \frac{p^{((n_x+n+1)\,d)}(y, y)}{p^{(nd)}(y, y)} \xrightarrow{n \to +\infty} \alpha_x \end{aligned}$$

Thus the sequence of functions $x \mapsto p^{(nd+k_x)}(x, y)/p^{(nd)}(y, y)$ is bounded pointwise in x, i.e., relatively compact with respect to pointwise convergence. Hence there exists a subsequence $\{n'\} \subset \mathbb{N}$ and a function g such that

$$\lim_{n'} \frac{p^{(n'd+k_x)}(x, y)}{p^{(n'd)}(y, y)} = g(x), \qquad \forall x \in X$$

The function g is nonnegative and superharmonic, in fact

$$Pg(x) = \sum_{w \in X} \lim_{n'} p(x, w) \frac{p^{(n'd+k_w)}(w, y)}{p^{(n'd)}(y, y)} \leqslant \lim_{n'} \frac{p^{(n'd+k_x)}(x, y)}{p^{(n'd)}(y, y)} = g(x)$$

using Fatou's Lemma and the fact that $p(x, w) \neq 0$ if and only if $k_w = k_x - 1 \pmod{d}$ (note that if $k_x = 0$ Eq. (2) is needed). By recurrence g is constant and, since $g(y) = 1$, we have

$$\lim_{n \to +\infty} \frac{p^{(nd+m)}(x, y)}{p^{(nd)}(y, y)} = 1 \quad \blacksquare \tag{3}$$

This result allows us to give a necessary and sufficent condition for a class of random walks to be asymptotically isotropic.

Theorem 3.2. (i) Let (X, P) be a recurrent, strongly periodic random walk of period d and let $x \in X$. Then (X, P) is asymptotically isotropic with respect to x if and only if for every $k \in \mathbb{N}$ and for all $y, y_1 \in S(x, k)$, we have

$$p^{(n)}(y, y) \sim^n p^{(n)}(y_1, y_1) \tag{4}$$

(ii) Let (X, P) be a reversible, recurrent, strongly periodic random walk of period d and let $x \in X$. Then (X, P) is asymptotically isotropic with respect to x if and only if the reversibility function v is constant on $S(x, k)$ for every $k \in \mathbb{N}$.

Proof. (i) Let $y, y_1 \in S(x, k)$ then $p^{(k)}(x, y) > 0$, $p^{(k)}(x, y_1) > 0$; by Theorem 3.1, $p^{(n+k)}(x, y) \sim^n p^{(n+k)}(x, y_1)$ if and only if $p^{(n)}(y, y) \sim^n p^{(n)}(y_1, y_1)$.

(ii) If v is the reversibility function of (X, P) and $y, y_1 \in X$ such that $d(y, y_1) = m$ then, by Theorem 3.1,

$$\begin{aligned}\frac{p^{(nd)}(y, y)}{p^{(nd)}(y_1, y_1)} &= \frac{p^{(nd)}(y, y)}{p^{(nd+m)}(y_1, y)} \cdot \frac{p^{(nd+m)}(y_1, y)}{p^{(nd)}(y_1, y_1)} \\ &= \frac{p^{(nd)}(y, y)}{p^{(nd+m)}(y_1, y)} \cdot \frac{v(y)\, p^{(nd+m)}(y, y_1)}{v(y_1)\, p^{(nd)}(y_1, y_1)} \xrightarrow{n \to +\infty} \frac{v(y)}{v(y_1)}\end{aligned} \tag{5}$$

Hence the first part of this theorem leads us to the conclusion. ∎

Remark 3.3. There are two situations in which Theorem 3.2 is easily verified: first of all if (X, P) is a recurrent random walk on a Cayley graph of a group; secondly if (X, P) is a symmetric (i.e., reversible with $v \equiv 1$), recurrent, strongly periodic random walk. In these cases (X, P) is asymptotically isotropic with respect to every point $x \in X$.

We want to point out that the properties of isotropy and asymptotical isotropy are closely linked with the transition probabilities; a graph X could be equipped with stochastic matrices P, P_1 such that the random walks (X, P) and (X, P_1) have a different behaviour. Explicit examples of this fact are the so called n-dimensional combs (see ref. 10). These are, roughly speaking, infinite trees recursively defined as follows: a one-dimensional comb (1-comb) is an infinite linear chain; an $(n+1)$-dimensional comb ($(n+1)$-comb) is obtained attaching a 1-comb to every point of an n-comb. By using techniques of electric networks on graphs (see ref. 9, Theorem 4.8) it is easy to show that the simple random walk on an n-comb is recurrent; moreover it is clearly reversible (with $v(x) = \deg(x)$) and strongly periodic of period 2 but we cannot choose any point $x \in X$ such that the reversibility function v is constant on every sphere $S(x, k)$; hence, by Theorem 3.2(ii), this random walk is not asymptotically isotropic. However there is a natural way to assign to a tree-like graph, with $\deg(\cdot) \geqslant 2$, a symmetric random walk; if such a graph is an n-comb this walk turns out to be recurrent, strongly periodic with period 2 and then, by Theorem 3.2(ii), also asymptotically isotropic with respect to any point $x \in X$.

In the next theorem we slightly extend a result of Avez (ref. 1, Theorem 1). We consider a random walk on a group; the walk is not required to be recurrent, nor the group to be finitely generated.

Theorem 3.4. Let G be an amenable, countable group and μ a symmetric, probability measure whose support generates G then

$$p^{(n+m)}(h, g) \sim^n p^{(n+k)}(h, g_1),$$

where $h, g, g_1 \in G$ and $m, k \in \mathbb{N}$ such that $p^{(m)}(h, g) > 0$ and $p^{(k)}(h, g_1) > 0$.

Proof. Since G is a group it is enough to consider only the case $h = e$, where e is the unit of G. If we prove Eq. (3) we obtain the thesis by using the same considerations as in Theorem 3.2 (Eq. (4) holds by Remark 3.3).

By our hypotheses we have that the period of the random walk, d, must be 1 or 2. If $d = 1$ Avez proved (3) in ref. 1, Theorem 1.

If $d = 2$ and m is even (i.e., $d(e, g)$ is even) then

$$\lim_{n \to +\infty} \frac{p^{(2n+m)}(e, g)}{p^{(2n)}(e, e)} = \lim_{n \to +\infty} \frac{p^{(2n+m)}(e, g)}{p^{(2n+m)}(e, e)} \cdot \frac{p^{(2n+m)}(e, e)}{p^{(2n+m-2)}(e, e)} \cdots \frac{p^{(2n+2)}(e, e)}{p^{(2n)}(e, e)} = 1 \tag{6}$$

(see ref. 1, Lemma 4 and Lemma 5). If $d = 2$ and m is odd (i.e., $d(e, g)$ is odd) then

$$\frac{p^{(2n+m)}(e, g)}{p^{(2n)}(e, e)} = \sum_{h \sim e} p(e, h) \frac{p^{(2n+m-1)}(e, h^{-1}g)}{p^{(2n)}(e, e)} \xrightarrow{n \to +\infty} \sum_{h \sim e} p(e, h) = 1$$

(note that for all $n \geqslant 1$, for all $h \sim e$, $p^{(2n+m-1)}(e, h^{-1}g) > 0$) where evaluating the limit we used the fact that, because of Lemma 1 and Lemma 2 of ref. 1, (note that $m-1$ is even),

$$p(e, h) \frac{p^{(2n+m-1)}(e, h^{-1}g)}{p^{(2n)}(e, e)} \leqslant p(e, h) \tag{7}$$

By the previous part of the proof, we have that the left hand side of Eq. (7) converges to $p(e, h)$ if $n \to \infty$ then we can apply the Lebesgue's bounded convergence theorem to the sum in Eq. (6). ∎

By using the previous result, every symmetric random walk on a countable amenable group is an asymptotically isotropic random walk; in particular every symmetric random walk on a countable, abelian group is asymptotically isotropic. For example, this holds for every symmetric random walk on $\mathbb{Z}^d$. Note that in the case of $\mathbb{Z}^2$ even the simple random walk is not isotropic (by Eq. (11) below).

4. ASYMPTOTIC BEHAVIOUR OF THE GREEN FUNCTION IN $\mathbb{Z}^2$

Given a random walk (X, P) the generating function of the transition probabilities (usually called the Green function) is defined as the following power series:

$$G_X(x, y \mid z) = \sum_{n=0}^{+\infty} p^{(n)}(x, y)\, z^n, \qquad x, y \in X, \quad z \in \mathbb{C} \tag{8}$$

and its radius of convergence will be denoted by r.

Observe that by mean of this generating function, Definition 1.1 is equivalent to

$$\forall k \in \mathbb{N}, \quad \forall y_1, y_2 \in S(x_0, k), \qquad G_X(x_0, y_1 \mid z) = G_X(x_0, y_2 \mid z), \quad \forall z : |z| < r \tag{9}$$

Our next step is to calculate the asymptotic value of the Green function of $\mathbb{Z}^2$ when the distance between the starting and the ending point grows up to infinity and z is a real number, $0 < z < 1$. The proof of the main result of this section (Theorem 4.5) exploits Laplace-type techniques, compare, e.g., with ref. 11, Theorem 3.1.

It is well known that $p^{(n)}(x, y) = 0$ if $d(x, y) > n$; from now on when we write $p^{(n)}(x, y)$ we mean $d(x, y)/n \in [0, 1]$. Note also that the Green function can be written as

$$G(x, y \mid z) = \sum_{n=d(x, y)}^{+\infty} p^{(n)}(x, y)\, z^n \tag{10}$$

In order to obtain a useful expression for the transition probabilities $p^{(n)}(x, y)$ of the simple random walk on $\mathbb{Z}^2$ we consider the direct product of two copies of the simple random walk on $\mathbb{Z}$ according to the following definition.

Definition 4.1. Let $(X_i, P_i)_{i=1,\dots,m}$ be a finite family of random walks; we call *direct product* of the family $(X_i, P_i)_{i=1,\dots,m}$ a random walk on the direct product $\prod_{i=1}^m X_i$ with transition probabilities

$$p^{(n)}((x_1,\dots, x_m), (y_1,\dots, y_m)) := \prod_{i=1}^{n} p_{X_i}^{(n)}(x_i, y_i)$$

In particular, if $m=2$, $X_1 = X_2 = \mathbb{Z}$ and $P_1 = P_2$ are the transition matrices of the simple random walk, let $(\mathbb{Z}^2, P_{\mathbb{Z}\times\mathbb{Z}})$ be the product and $\{p_{\mathbb{Z}\times\mathbb{Z}}((x_1, x_2), (y_1, y_2)) : x_i, y_i \in \mathbb{Z}, i = 1, 2\}$ be the transition probabilities.

As a consequence of the periodicity of the simple random walk on $\mathbb{Z}$, the random walk $(\mathbb{Z}^2, P_{\mathbb{Z}\times\mathbb{Z}})$, starting from $(0, 0)$ may reach only the points in $\mathbb{Z}^2$ such that $x + y$ is even, while the same random walk, starting from a point $(\bar{x}, \bar{y})$ such that $\bar{x} + \bar{y}$ is odd may reach only points with the same property. In this way $\mathbb{Z}^2$ turns out to be partitioned into two subsets, and each of them is isomorphic to $\mathbb{Z}^2$ itself.

Consider the following isomorphism between $\mathbb{Z}^2$ and its subset associated to the random walk $(\mathbb{Z}^2, P_{\mathbb{Z}\times\mathbb{Z}})$ starting from the origin:

$$\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\} \mapsto \{(a, b) \in \mathbb{Z}^2 : a + b \text{ is even}\}$$

$$(x, y) \mapsto (x - y, x + y)$$

It is obvious that

$$\begin{aligned} p_{\mathbb{Z}^2}^{(n)}((0, 0), (x, y)) &= p_{\mathbb{Z}\times\mathbb{Z}}^{(n)}((0, 0), (x - y, x + y)) \\ &= p_{\mathbb{Z}}^{(n)}(0, x - y)\, p_{\mathbb{Z}}^{(n)}(0, x + y) \end{aligned}$$

Recalling that the period of the simple random walk on $\mathbb{Z}$ is 2, the last product can be written as

$$p_{\mathbb{Z}^2}^{(n)}((0, 0), (x, y)) = \begin{cases} \dfrac{1}{2^{2n}} \dbinom{n}{(n + |x - y|)/2} \dbinom{n}{(n + |x + y|)/2} & \text{if } n + |x| + |y| \text{ is even} \\ 0 & \text{if } n + |x| + |y| \text{ is odd} \end{cases} \tag{11}$$

Now we are interested in some kind of asymptotic estimate of these transition probabilities when n goes to infinity; the next theorems will provide us this information.

Theorem 4.2. The transition probabilities $p_{\mathbb{Z}}^{(n)}(0, k)$ can be written as

$$p_{\mathbb{Z}}^{(n)}(0, k) = \begin{cases} \beta_n\left(\dfrac{|k|}{n}\right) \exp\left\{n\varphi\left(\dfrac{|k|}{n}\right)\right\} & \text{if } n+|k| \text{ is even} \\ 0 & \text{if } n+|k| \text{ is odd} \end{cases} \tag{12}$$

where

$$\varphi(\xi) = \frac{1}{2}\left(\xi \log\left(\frac{1-\xi}{1+\xi}\right) - \log(1-\xi^2)\right) \tag{13}$$

$$\lim_{n \to \infty} \frac{1}{n} \sup_{\xi \in [0, 1]} |\log(\beta_n(\xi))| = 0 \tag{14}$$

and the estimate

$$\beta_n(\xi) \sim^n \begin{cases} \sqrt{\dfrac{2}{\pi n(1-\xi^2)}} & \text{if } \xi \in [0, 1-n^{-3/4}] \\ \left[\dfrac{n(1-\xi)}{2e}\right]^{(n/2)(1-\xi)} \cdot \sqrt{\dfrac{2}{1+\xi}} \Big/ \left[\dfrac{n}{2}(1-\xi)\right]! & \text{if } \xi \in (1-n^{-3/4}, 1] \end{cases} \tag{15}$$

holds uniformly with respect to ξ.

Proof. Eqs. (12), (13), (15) follow applying De Moivre–Stirling's formula $[n! = n^n \sqrt{2\pi n} \exp(-n + \theta_n/12n),\ |\theta_n| \leqslant 1]$ to Eq. (11); Eq. (14) follows noting that the property holds if ξ is allowed to vary among $\{0, 1/n, 2/n, ..., 1\}$, which are the only values of interest. ∎

We remark that the exponent $-3/4$ has no particular meaning, instead of it we could have chosen any value between -1 and 0.

By using Eq. (11) and the previous theorem we can obtain the following corollary.

Corollary 4.3. The transition probabilities $p_{\mathbb{Z}^2}^{(n)}(e, (x, y))$ (where $e = (0, 0)$) can be written as

$$p_{\mathbb{Z}^2}^{(n)}(e, (x, y)) = \begin{cases} \beta_n\left(\dfrac{|x-y|}{n}\right) \beta_n\left(\dfrac{|x+y|}{n}\right) \exp\left\{n\varphi\left(\dfrac{|x-y|}{n}\right)\right\} \exp\left\{n\varphi\left(\dfrac{|x+y|}{n}\right)\right\} \\ \qquad \text{if } n+|x|+|y| \text{ is even} \\ 0 \qquad \text{if } n+|x|+|y| \text{ is odd} \end{cases} \tag{16}$$

where $\varphi(\xi)$ is defined by Eq. (13) and

$$\beta_n(\xi_1)\,\beta_n(\xi_2) \sim^n \frac{2}{\pi n\,\sqrt{(1-\xi_1^2)(1-\xi_2^2)}} \tag{17}$$

the estimate being uniform with respect to $\xi_1, \xi_2 \in [0, 1-n^{-3/4}]$.

Our aim is now to understand when we can replace each term in the sum

$$G_{\mathbb{Z}^2}(x, y \mid z) = \sum_{n=|x|+|y|}^{+\infty} p^{(n)}_{\mathbb{Z}^2}(x, y)\, z^n \tag{18}$$

by its asymptotic value; the next (technical) lemma gives us an answer.

Lemma 4.4. Let $\{a_n(k)\}_n$ and $\{b_n(k)\}_n$ be two families of sequences whose sign does not depend on n such that $a_n(k) = b_n(k)(1+o_n(k))$. Let $\{A_n\}_n$ a sequence of subsets of $\mathbb{N}$, such that for all $\varepsilon > 0$ there exists n_ε satisfying $|o_n(k)| < \varepsilon$, for all $n \geqslant n_\varepsilon$, $k \in A_n$. If $\{r_k\}_k$ is a real sequence such that $r_k \xrightarrow{k \to \infty} +\infty$ and $|\sum_{n \geqslant r_k :\, A_n \ni k} a_n(k)| < +\infty$ for all $k \in \mathbb{N}$, then

$$\sum_{n \geqslant r_k :\, A_n \ni k} a_n(k) \sim^k \sum_{n \geqslant r_k :\, A_n \ni k} b_n(k)$$

Proof. Let us fix $\varepsilon \in (0, 1)$ and $k_\varepsilon \in \mathbb{N}$ such that for all $k \geqslant k_\varepsilon$ we have that $r_k > n_\varepsilon$ then

$$(1-\varepsilon)\left|\sum_{n \geqslant r_k :\, A_n \ni k} b_n(k)\right| \leqslant \left|\sum_{n \geqslant r_k :\, A_n \ni k} a_n(k)\right| \leqslant (1+\varepsilon)\left|\sum_{n \geqslant r_k :\, A_n \ni k} b_n(k)\right|$$

If $(x, y) \in \mathbb{Z}^2 \setminus \{e\}$, we define $\lambda = |(|x|-|y|)/(|x|+|y|)|$ and let

$$\begin{aligned}\Phi(\xi) \equiv \Phi(\xi, z, \lambda) := \frac{1}{\xi}\Bigg[\log z + \frac{1}{2}\Bigg(\xi \log \frac{1-\xi}{1+\xi} - \log(1-\xi^2) \\ + \lambda\xi \log \frac{1-\lambda\xi}{1+\lambda\xi} - \log(1-\lambda^2\xi^2)\Bigg)\Bigg]\end{aligned} \tag{19}$$

It is easy to prove that $\xi \mapsto \Phi(\xi)$ attains only one maximum in the interval $(0, 1)$, namely in

$$\xi(z) \equiv \xi(z, \lambda) := \sqrt{\frac{2(1-z^2)}{1+\lambda^2+\sqrt{(1-\lambda^2)^2+4z^2\lambda^2}}} \tag{20}$$

In particular it is easy to show that $\Phi(\xi(z))$ can be written as

$$\Phi(\xi(z)) \equiv \Phi(\xi(z,\lambda),\lambda) = \frac{1}{2}\left[\log\frac{1-\xi(z)}{1+\xi(z)} + \lambda\log\frac{1-\lambda\xi(z)}{1+\lambda\xi(z)}\right] \tag{21}$$

Theorem 4.5. The Green function of the simple random walk on $\mathbb{Z}^2$ admits the following asymptotic estimate:

$$G_{\mathbb{Z}^2}(e, k(x,y)\,|\,z) \sim^k \frac{\exp\{k(|x|+|y|)\,\Phi(\xi(z))\}}{\xi(z)\sqrt{k(|x|+|y|)}\sqrt{1-\xi(z)^2}\sqrt{1-\lambda^2\xi(z)^2}}\sqrt{\frac{-2}{\pi\Phi''(\xi(z))}} \tag{22}$$

where $(x,y)\in\mathbb{Z}^2\setminus\{e\}$ and $k(x,y)=(kx,ky)$.

Proof. We first note that, taking into account the period of the simple random walk in $\mathbb{Z}^2$, we can write the Green function as follows:

$$G_{\mathbb{Z}^2}(e, k(x,y)\,|\,z) := \sum_{n=0}^{\infty} p_{\mathbb{Z}^2}^{(2n+k(|x|+|y|))}(e, k(x,y))\, z^{2n+k(|x|+|y|)} \tag{23}$$

We split the sum (23) into two parts:

$$G_{\mathbb{Z}^2}(e, k(x,y)\,|\,z) = S_1(k(x,y)) + S_2(k(x,y))$$

where

$$S_1(k(x,y)) = \sum_{n\,:\,|\xi_n-\xi(z)|<\delta} p_{\mathbb{Z}^2}^{(2n+k(|x|+|y|))}(e, k(x,y))\, z^{2n+k(|x|+|y|)} \tag{24}$$

and $S_2(k(x,y))$ is the rest. Moreover, in Eq. (24), we set $\xi_n = [k(|x|+|y|)]/[2n+k(|x|+|y|)]$, and we choose $\delta>0$ such that $I_\delta := (\xi(z)-\delta,\ \xi(z)+\delta)\subset[0,1]$ and

$$\begin{aligned} 0 &\geqslant \Phi(\xi)-\Phi(\xi(z)) = \tfrac{1}{2}\Phi''(\xi(z))(\xi-\xi(z))^2 + o((\xi-\xi(z))^2) \\ &\geqslant \tfrac{1}{4}\Phi''(\xi(z))(\xi-\xi(z))^2, \qquad \forall\xi\in I_\delta \end{aligned} \tag{25}$$

This choice allows us to apply Lemma 4.4 in the sum (24) by using the asymptotic estimates (16) and (17) to obtain

$$S_1(k(x,y)) \sim^k \sum_{n\,:\,|\xi_n-\xi(z)|<\delta} \frac{2\exp\{k(|x|+|y|)\,\Phi(\xi_n)\}}{\pi(2n+k(|x|+|y|))\sqrt{(1-\xi_n^2)(1-\lambda^2\xi_n^2)}} \tag{26}$$

since $|x+y|/(|x|+|y|), |x-y|/(|x|+|y|) \in \{1, \lambda\}$. We now perform a change of variable by setting $\sigma_j = (\xi_j - \xi(z))\sqrt{k(|x|+|y|)}$ and $\Delta\sigma_j = \sigma_j - \sigma_{j+1}$. Then $\Delta\sigma_j \sim^j (2k(|x|+|y|)^{3/2})/(2j+k(|x|+|y|))^2 = 2\xi_j^2/\sqrt{k(|x|+|y|)}$. Hence

$$
\begin{aligned}
S_1(k(x,y)) \sim^k & \sum_{n\,:\,|\sigma_n|<\delta\sqrt{k(|x|+|y|)}} \frac{\exp\{k(|x|+|y|)\,\Phi(\xi(z))\}}{\pi\sqrt{k(|x|+|y|)}} \\
& \times \frac{1}{\sqrt{1-\left(\xi(z)+\frac{\sigma_n}{\sqrt{k(|x|+|y|)}}\right)^2}} \\
& \times \frac{1}{\sqrt{1-\lambda^2\left(\xi(z)+\frac{\sigma_n}{\sqrt{k(|x|+|y|)}}\right)^2}} \\
& \times \frac{1}{\xi(z)+\frac{\sigma_n}{\sqrt{k(|x|+|y|)}}} \\
& \times \exp\left\{\frac{1}{2}\,\Phi''(\xi(z))\,\sigma_n^2 + o(\sigma_n^2)\right\}\Delta\sigma_n
\end{aligned}
\tag{27}
$$

Because of the estimate (25), we can apply Lebesgue's bounded convergence theorem to the Cauchy–Riemann sum in Eq. (27) obtaining

$$
\begin{aligned}
S_1(k(x,y)) \sim^k & \frac{\exp\{k(|x|+|y|)\,\Phi(\xi(z))\}}{\pi\xi(z)\sqrt{k(|x|+|y|)}\sqrt{1-\xi(z)^2}\sqrt{1-\lambda^2\xi(z)^2}} \\
& \times \int_{\mathbb{R}} \exp\left\{\frac{1}{2}\,\Phi''(\xi(z))\,\sigma^2\right\} d\sigma
\end{aligned}
\tag{28}
$$

The proof will be complete if we show that $S_2(k(x,y))/S_1(k(x,y)) \to 0$ as $k \to +\infty$, where

$$
S_2(k(x,y)) = \sum_{n\,:\,|\xi_n-\xi(z)|\geqslant\delta} p_{\mathbb{Z}^2}^{(2n+k(|x|+|y|))}(e, k(x,y))\, z^{2n+k(|x|+|y|)} \tag{29}
$$

Using Eq. (28), we obtain that

$$
\frac{S_2(k(x,y))}{S_1(k(x,y))} \sim^k C S_2(k(x,y))\sqrt{k(|x|+|y|)}\exp\{-k(|x|+|y|)\,\Phi(\xi(z))\} \tag{30}
$$

We note that there exists an $\varepsilon_1 > 0$ such that

$$\Phi(\xi) - \Phi(\xi(z)) \leqslant -\varepsilon_1, \qquad \forall \xi : |\xi - \xi(z)| \geqslant \delta$$

Elementary computations show that

$$\beta_n(\xi) \in \left[\sqrt{\frac{1}{\pi n}} \exp\left\{ -1 - \frac{1}{4n} \right\}, \sqrt{\frac{2}{\pi}} \exp\left\{ 1 + \frac{1}{4n} \right\} \right]$$

if $\xi = k/n$, $k = 0, 1,..., n-1$, while $\beta_n(1) = 1$. Hence

$$\begin{aligned} &S_2(k(x, y)) \exp\{ -k(|x| + |y|) \, \Phi(\xi(z)) \} \\ &\quad \leqslant \sum_{n=0}^{+\infty} C' \exp\{ (\Phi(\xi_n) - \Phi(\xi(z))) \, k(|x| + |y|) \} \end{aligned}$$

where one could show that it is possible to apply Lebesgue's bounded convergence theorem. This implies that the series tends to zero as k tends to infinity, which with Eq. (30) leads to the conclusion. ∎

In the context of the simple random walk, it appears that the above combinatorial method is more efficient than the use of Fourier transformation as in ref. 12.

Remark 4.6. The main term in the Eq. (22) is the exponential of $k(|x| + |y|) \, \Phi(\xi(z))$; in order to compare the asymptotic behaviour of the Green function along different directions (i.e., corresponding to different values of λ) it is useful to study the function

$$\lambda \mapsto \Phi(\xi(z, \lambda), \lambda) \tag{31}$$

with $\lambda \in [0, 1]$. Obviously by definition λ must be rational, but we can look at it as a real variable to study Eq. (31). By differentiating Eq. (19) and taking into account that $\partial_\xi \Phi(\xi(z), \lambda) = 0$ (since $\xi(z)$ is a maximum), we have that $(d/d\lambda) \, \Phi(\xi(z), \lambda) = \partial_\lambda \Phi(\xi(z), \lambda) < 0$, for all $z \in (0, 1)$. This means that if we choose (x, y) and $(\bar{x}, \bar{y})$ with the same (strictly positive) distance from the origin and so that $\lambda < \bar{\lambda}$ (where λ is the same as before and $\bar{\lambda} := |(|\bar{x}| - |\bar{y}|)/(|\bar{x}| + |\bar{y}|)|$) then

$$\lim_{k \to +\infty} \frac{G_{\mathbb{Z}^2}(e, k(x, y) \mid z)}{G_{\mathbb{Z}^2}(e, k(\bar{x}, \bar{y}) \mid z)} = +\infty \tag{32}$$

We recall that $\lambda = 1$ refers to $k(x, y)$ on one of the axes, whereas $\lambda = 0$ refers to $k(x, y)$ on one of the bisectors.

5. THE ANISOTROPY OF RANDOM WALKS ON $\Gamma * \mathbb{Z}^2$

In this paragraph we want to discuss a particular random walk on the Cayley graph of a free product of groups.

Given a (finite or countable) family $(\Gamma_i)_{i \in I}$ of random walks, where each Γ_i is a finitely generated, discrete group, with $\bigcap_{i \in I} \Gamma_i = \{e\}$, the unit element, and μ_i is an irreducible probability measure defined on Γ_i, $i \in I$, we define the free product of these random walks to be the random walk (Γ, μ) where $\Gamma = *_{i \in I} \Gamma_i$ is the free product of the Γ_i's and μ is a convex combination of the μ_i's:

$$\mu = \sum_{i \in I} a_i \mu_i, \quad a_i > 0, \qquad \sum_{i \in I} a_i = 1$$

(for further details see, e.g., ref. 9, Paragraph 6.D). Of course the free product depends on the choice of the weights a_i, $i \in I$.

We already introduced the Green function $G_X(x, y \mid z)$ of a random walk (X, P) as a power series defined by Eq. (8). In addition we need another important generating function related to the Green function; given a general Markov chain Z_n described by the random walk (X, P), we set the first return probabilities and their generating function

$$f^{(n)}(x, y) = \mathbf{Pr}[Z_n = y, Z_k \neq y, k = 1,..., n-1 \mid Z_0 = x]$$

$$f^{(0)}(x, y) = 0$$

$$F_X(x, y \mid z) = \sum_{n=0}^{+\infty} f^{(n)}(x, y)\, z^n, \qquad x, y \in X, \quad z \in \mathbb{C}$$

In their common domain of convergence

$$G_X(x, y \mid z) = F_X(x, y \mid z)\, G_X(y, y \mid z) \tag{33}$$

where $x, y \in X$, $x \neq y$.

Let us now fix a root $o \in X$ (if X is a group, the natural choice for o is the unit element e) and denote by $\Phi_X(t)$ the analytic function defined in ref. 2, Paragraph 2 satisfying

$$\Phi_X(zG_X(o, o \mid z)) = G_X(o, o \mid z), \qquad z \in \mathbb{C}.$$

Let $\Psi_X(t) = \Phi_X(t) - t\Phi'_X(t)$, $0 < t < r$; the results in ref. 2 show that there exists a unique $\theta_X \in (0, +\infty]$ such that

$$\lim_{t \to \theta_X^-} \Psi_X(t) = 0$$

It is easy to show that the function $t \mapsto t/\Phi_X(t)$ is strictly increasing in $(0, \theta_X)$ and

$$\lim_{t \to \theta_X^-} \frac{t}{\Phi_X(t)} \equiv \sup_{t \in (0, \theta_X)} \frac{t}{\Phi_X(t)} = r \tag{34}$$

Let us define, following ref. 2, Paragraph 3

$$\Phi_X(x \mid t) = F_X\left(o, x \,\middle|\, \frac{t}{\Phi_X(t)}\right)$$

where $x \in \Gamma \backslash \{o\}$; we note that if $t \in (0, \theta_X)$ then the previous equation is well defined since $t/\Phi_X(t) \leqslant r$.

Let us take a (non trivial) discrete group Γ and a probability measure μ_1 defined on it. Let $(\mathbb{Z}^2, \mu_2)$ be the simple random walk on $\mathbb{Z}^2$ and consider the free product $(\Gamma * \mathbb{Z}^2, \mu)$ where $\mu = a_1\mu_1 + a_2\mu_2$ according to the previous definition.

Let us denote by G_i, F_i, r_i, Φ_i, Ψ_i, θ_i, $i = 1, 2, *$, the functions and quantities defined before, referring to (Γ, μ_1) if $i = 1$, to $(\mathbb{Z}^2, \mu_2)$ if $i = 2$ and to the free product if $i = *$ (for numerical estimates of θ for simple random walk on $\mathbb{Z}^d$ see ref. 4).

It is well known (see ref. 13) that a free product $\Gamma = *_{i \in I} \Gamma_i$ is amenable if and only if $|I| = 2$ and $|\Gamma_1| = |\Gamma_2| = 2$ (this case is treated in refs. 14 and 15). This is not our case, since $\Gamma_2 = \mathbb{Z}^2$, therefore $(\Gamma * \mathbb{Z}^2, \mu)$ is nonamenable, hence $r_* > 1$, $\theta_* < +\infty$ and the walk is not r-recurrent (see ref. 16). Let d be the period of this random walk: obviously, $d = 2$ if the period of (Γ, μ_1) is even, and $d = 1$ otherwise. From now on let the following assumption hold.

Assumption 5.1. One of the following holds:

(i) (Γ, μ_1) is an r-recurrent and reversible random walk;

(ii) $\Gamma = \mathbb{Z}^n$, with $n \leqslant 4$ and μ_1 has finite exponential moments, or zero mean and finite moments of order $\min\{n, 2\}$;

(iii) Γ has polynomial growth with degree $\alpha \leqslant 4$ and μ_1 is symmetric with finite moments of order $\min\{\alpha, 2\}$.

Let us note that the simple random walk on $\mathbb{Z}^2$ satisfies all the previous conditions. If (Γ, μ_1) satisfies (i) (resp. (ii) or (iii)) we have that for $(\Gamma * \mathbb{Z}^2, \mu)$ Theorem 6 in ref. 2 (resp. Corollary 17.7 in ref. 17) holds.

Theorem 9.19 in ref. 17 shows that $\theta_* \leqslant \min\{\theta_1/a_1, \theta_2/a_2\}$. Moreover, we note that if $t \in (0, \theta_*)$ then $a_2 t/\Phi_2(a_2 t) \in (0, 1)$ because of Eq. (34) and $r_2 = 1$.

From now on, $\{k(x, y)\}_k$ and $\{k(\bar{x}, \bar{y})\}_k$ will be two families of "words" belonging to the subgroup $\{e\} * \mathbb{Z}^2$ of $\Gamma * \mathbb{Z}^2$.

Theorem 5.2. Suppose that (Γ, μ_1) satisfy Assumption 5.1; let $|x| + |y| = |\bar{x}| + |\bar{y}| \neq 0$ and $\lambda < \bar{\lambda}$ (where $\lambda = |(|x| - |y|)/(|x| + |y|)|$ and $\bar{\lambda} = |(|\bar{x}| - |\bar{y}|)/(|\bar{x}| + |\bar{y}|)|$) then there exists a function $g((x, y), (\bar{x}, \bar{y}), k)$ such that

$$\frac{p_{\Gamma * \mathbb{Z}^2}^{(nd + k(|x| + |y|))}(e, k(x, y))}{p_{\Gamma * \mathbb{Z}^2}^{(nd + k(|\bar{x}| + |\bar{y}|))}(e, k(\bar{x}, \bar{y}))} \sim^n g((x, y), (\bar{x}, \bar{y}), k) \tag{35}$$

where d is the period of the random walk on the free product, and

$$\lim_{k \to +\infty} g((x, y), (\bar{x}, \bar{y}), k) = +\infty \tag{36}$$

Proof. As shown in ref. 2, Theorem 6 or in ref. 17, Corollary 17.7, we have that

$$\frac{p_{\Gamma * \mathbb{Z}^2}^{(nd + k(|x| + |y|))}(e, k(x, y))}{p_{\Gamma * \mathbb{Z}^2}^{(nd + k(|\bar{x}| + |\bar{y}|))}(e, k(\bar{x}, \bar{y}))} \sim^n \frac{\Phi_2(k(x, y) \mid a_2\theta_*)}{\Phi_2(k(\bar{x}, \bar{y}) \mid a_2\theta_*)} \cdot \frac{\dfrac{1}{\theta_*} + \dfrac{a_2 \Phi_2'(k(x, y) \mid a_2\theta_*)}{\Phi_2(k(x, y) \mid a_2\theta_*)}}{\dfrac{1}{\theta_*} + \dfrac{a_2 \Phi_2'(k(\bar{x}, \bar{y}) \mid a_2\theta_*)}{\Phi_2(k(\bar{x}, \bar{y}) \mid a_2\theta_*)}} \tag{37}$$

In order to study the previous equation, we need to know F_2 and F_2'. Assuming from now on $t \in \mathbb{R}$, $t > 0$

$$\frac{F_2'(e, k(x, y) \mid t)}{F_2(e, k(x, y) \mid t)} = \frac{k}{t} + \frac{1}{t} \frac{\sum_{i=1}^{+\infty} \sum_{n=k+i}^{+\infty} f^{(n)}(e, k(x, y))\, t^n}{\sum_{n=k}^{+\infty} f^{(n)}(e, k(x, y))\, t^n} \geqslant \frac{k}{t} \xrightarrow{k \to +\infty} +\infty$$

and using Eq. (33) (recalling that, on a group X, $G_X(y, y \mid z) = G_X(e, e \mid z)$ for every $y \in X$) we have that

$$\frac{F_2'(e, k(x, y) \mid t)}{F_2(e, k(x, y) \mid t)} \sim^k \frac{G_2'(e, k(x, y) \mid t)}{G_2(e, k(x, y) \mid t)} \tag{38}$$

Performing similar computations as in Theorem 4.5, using

$$G_2'(e, k(x, y) \mid t)$$
$$= \frac{1}{t} \sum_{n=0}^{+\infty} (nd + k(|x| + |y|))\, p^{(nd + k(|x| + |y|))}(e, k(x, y))\, t^{nd + k(|x| + |y|)} \tag{39}$$

we obtain

$$G_2'(e,k(x,y)\,|\,t) \sim^k \frac{\sqrt{k(|x|+|y|)}\exp\{k(|x|+|y|)\,\Phi(\xi(t))\}}{t\xi(t)^2\sqrt{1-\xi(t)^2}\sqrt{1-\lambda^2\xi(t)^2}}\sqrt{\frac{-2}{\pi\Phi''(\xi(t))}} \tag{40}$$

Obviously, Eqs. (38) and (40) hold for every $(x, y) \in \mathbb{Z}^2 \setminus \{0, 0\}$. Now, using Eqs. (22), (37), (38), and (40), it follows that

$$\frac{p_{\Gamma*\mathbb{Z}^2}^{(nd+k(|x|+|y|))}(e,k(x,y))}{p_{\Gamma*\mathbb{Z}^2}^{(nd+k(|\bar{x}|+|\bar{y}|))}(e,k(\bar{x},\bar{y}))} \sim^n \frac{G_2\left(e,k(x,y)\,\middle|\,\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)}\right)}{G_2\left(e,k(\bar{x},\bar{y})\,\middle|\,\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)}\right)} \cdot \frac{\xi\left(\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)},\bar{\lambda}\right)}{\xi\left(\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)},\lambda\right)}$$

where ξ is the function defined by Eq. (20). Finally, Eq. (32) implies that

$$\lim_{k\to+\infty} \frac{G_2\left(e,k(x,y)\,\middle|\,\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)}\right)}{G_2\left(e,k(\bar{x},\bar{y})\,\middle|\,\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)}\right)} \cdot \frac{\xi\left(\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*)},\bar{\lambda}\right)}{\xi\left(\dfrac{a_2\theta_*}{\Phi_2(a_2\theta_*},\lambda\right)} = +\infty \quad \blacksquare$$

It is obvious that if $(\Gamma * \mathbb{Z}^2, \mu)$ were an asymptotically isotropic random walk then $g((x, y), (\bar{x}, \bar{y}), k) \equiv 1$; if the hypotheses of Theorem 5.2 are satisfied then Eq. (36) shows that $(\Gamma * \mathbb{Z}^2, \mu)$ is not an asymptotically isotropic random walk.

We obtain the simple random walk on $\mathbb{Z} * \mathbb{Z}^2$ setting $\Gamma = \mathbb{Z}$, μ_1 the simple random walk on $\mathbb{Z}$, $a_1 = 1/3$ and $a_2 = 2/3$. Obviously $(\mathbb{Z}, \mu_1)$ satisfies Assumption 5.1, then by Theorem 5.2 the simple random walk on the free product is far from being asymptotically isotropic, hence it cannot be isotropic as was supposed in ref. 5 Theorem 3.1.

Moreover ref. 5, Eq. (3.1) would imply that the radius of convergence of this random walk should be $\exp(1/6)$, whereas we performed numerical computations with MathCad finding an approximate value $r_* \approx 1.231$.

6. DISCUSSION

As we showed in this paper, the property of isotropy is strongly linked to geometrical properties of the graph, more than the asymptotical isotropy. The main property we used is, roughly speaking, the rotational symmetry of both the graph and the random walk; this is the substantial meaning of the x_0-radiality and Γ_{x_0}-invariance.

The point (ii) of Theorem 2.2 is a partial converse of the point (i) in the case of trees: we showed that every random walk which is isotropic with respect to a point x_0 needs to be invariant with respect to all the rotations centered in x_0. The question whether a tree equipped with an x_0-isotropic random walk must be x_0-radial is (as far as we know) still open. Besides, we do not know if there is an equivalent of Theorem 2.2(ii) for general graphs.

The property of asymptotical isotropy does not depend so much on geometrical properties of the graph. The main result (Theorem 3.1) shows that, upender the hypoteses of recurrence and strong periodicity, the random walk "forgets" the starting point and, as time grows up to infinity, it depends only on the ending point. This has many consequences as we showed in Theorem 3.2. Another important consequence, which derives by Eq. (5), is that the asymptotical values for the transition probabilities of a reversible, recurrent, strongly periodic random walk (X, P) are independent of the starting points and depend only on the value of the reversibility function ν evaluated in the ending points. More precisely it is easy to show, from Eq. (5), that, if $x, y, x_1, y_1 \in X$ and $m, k \in \mathbb{N}$ are such that $p^{(m)}(x, y) > 0$ and $p^{(k)}(x_1, y_1) > 0$, then

$$p^{(n+m)}(x, y) \sim^n p^{(n+k)}(x_1, y_1) \frac{\nu(y)}{\nu(y_1)}$$

This means that if we know the asymptotical value of the transition probabilities for a couple (x, y) we know them also for the other pairs.

The family of examples of non asymptotically isotropic random walks, that we gave in Paragraph 5, is strongly linked with a property of the Green function of $\mathbb{Z}^2$ (see Eq. (32)). This is true because we chose to compare two paths lying, roughly speaking, on the copy of $\mathbb{Z}^2$ which contains the unit of the group e. This behaviour is quite singular; in fact the simple random walk on $\mathbb{Z}^2$ is asymptotically isotropic; we proved that if we consider the free product between $\mathbb{Z}^2$ and a group Γ (which satisfies some assumptions), the free product of the random walks behaves in a totally different way. Suppose that the number of steps n is large and consider the probability of the random walk to be in a certain point belonging to $\{e\} * \mathbb{Z}^2$: the closer the point is to one of the bisectors of $\mathbb{Z}^2$, the higher is this probability.

The strategy we used here is rather general and could be employed to construct other classes of examples (it suffices to find a group playing the same role as $\mathbb{Z}^2$, with a Green function behaving in a similar way (see Eq. (32)).

In particular we showed that the simple random walk on $\mathbb{Z} * \mathbb{Z}^2$ is not asymptotically isotropic; since this group is a subgroup of every locally free group $\mathscr{L}\mathscr{F}_n(d)$, provided that $n \geqslant d+2$, (see ref. 5, Definition 3) one could conjecture that none of the simple random walks on these groups is isotropic.

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